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# On the sum of sine products

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**Abstract**

Given $N \geqslant 1$ and $d \geqslant 1$, we show the existence of positive integers $\lambda_{kj}$ for $1 \leqslant k \leqslant N$ and $1 \leqslant j \leqslant d$ such that

$$\left|\sum_{k=1}^{N}\prod_{j=1}^{d}\sin\lambda_{kj}x_j\right| \leqslant CN^{(d+1)/(2d+1)}$$

for all real $x_1,\ldots,x_N$, where $C$ is a constant that depends only on $d$. This extends a result of Bourgain for the case $d=1$.


*Keywords:* Trigonometric polynomial; Sums of sines; Random variable; Probability measure

## 1. Introduction

Bourgain [1] has shown the existence of positive integers $\lambda_k$ such that

$$\left|\sum_{k=1}^{N}\sin\lambda_k x\right| \leqslant CN^{2/3}$$

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for all real $x$, where $C$ is an absolute constant. On the other hand, the trivial lower bound $CN^{1/2}$ cannot be achieved since Konyagin [2] has shown that for $N$ sufficiently large, and any set of distinct positive integers $\lambda_1, \ldots, \lambda_N$ there exits an $x$ such that

$$0.15\sqrt{\frac{N\log N}{\log\log N}} \leqslant \sum_{k=1}^{N} \sin\lambda_k x.$$

In this paper we extend Bourgain's result to higher dimensions. We will show that there exist positive integers $\lambda_{kj}$ for $1 \leqslant k \leqslant N$ and $1 \leqslant j \leqslant d$ such that

$$\left|\sum_{k=1}^{N}\prod_{j=1}^{d}\sin\lambda_{kj}x_j\right| \leqslant CN^{(d+1)/(2d+1)},$$

where $C$ depends only on $d$. Questions concerning bounds on sums of sines are delicate. For example it can be shown that if $\lambda_k$ grows polynomially or exponentially in $k$ then $\|\sum_{k=1}^{N}\sin\lambda_k x\|_\infty = O(N)$. The (random) construction in this paper produces $\lambda_k$ which grow like $\exp(k^{1/3})$ to give $\|\sum_{k=1}^{N}\sin\lambda_k x\|_\infty = O(N^{2/3})$. On the other hand, mere gap conditions on the $\lambda_k$ are not enough. It can be shown that if $\|\sum_{k=1}^{N}\sin\lambda_k x\|_\infty = O(N^{\delta})$ with $\delta < 1$ then there exists a set of $\epsilon_k = \pm 1$ such that $\|\sum_{k=1}^{N}\sin(\lambda_k+\epsilon_k)x\|_\infty = O(N)$. These questions regarding bounds on the supremum norm of sine sums are related to and to some extent motivated by questions regarding bounds on the norm of the Hilbert transform on finite-dimensional translation invariant spaces of continuous functions. More specifically, the question is: how large can the Hilbert transform be on a $2N$-dimensional translation invariant space of continuous functions? If the domain of the functions is the real line, then Bourgain's result shows that it can be $O(N^{1/3})$. If the domain of the functions is $d$-dimensional Euclidean space, then our result shows that the norm of the Hilbert transform can be $O(N^{d/(2d+1)})$. The authors are grateful to the referee for helpful comments and a careful reading of the manuscript.

## 2. Proof of the main theorem

In order to make this paper as self-contained as possible, we have chosen to give (the short) proofs of several well-known lemmas. The following three lemmas are well known.

**Lemma 2.1.** *Let $X$ be a random variable with mean $\mu$ satisfying $|x| \leqslant 1$. For any real $\lambda$*

$$E\big(e^{\lambda(X-\mu)}\big) \leqslant e^{\lambda^2/2}.$$

**Proof.** Let

$$\phi(\lambda) = \log E\big(e^{\lambda(X-\mu)}\big) = \log E[e^{\lambda X}e^{-\lambda\mu}] = \log E(e^{\lambda X}) - \lambda\mu.$$

It follows that

$$\phi'(z) = \frac{E(Xe^{zX})}{E(e^{zX})} - \mu$$

and

$$\phi''(z) = \frac{E(e^{zX})E(X^2e^{zX}) - [E(Xe^{zX})]^2}{[E(e^{zX})]^2} \leqslant \frac{E(X^2e^{zX})}{E(e^{zX})} \leqslant 1$$

since $|X| \leqslant 1$. Also $\phi(0) = 0$ and $\phi'(0) = 0$, and we have

$$\phi'(z) = \int_0^z \phi''(y)\,dy \leqslant \int_0^z dy = z,$$

$$\phi(\lambda) = \int_0^\lambda \phi'(z)\,dz \leqslant \int_0^\lambda z\,dz = \frac{\lambda^2}{2}.$$

It follows that $E(e^{\lambda(X-\mu)}) = e^{\phi(\lambda)} \leqslant e^{\lambda^2/2}$ and the proof of the lemma is therefore complete. □

**Lemma 2.2.** *Let $a \geqslant 0$, $\lambda \in \mathbf{R}$, $X : \Omega \to \mathbf{R}$ be a random variable. Then*

(a) $\mathcal{P}(X \geqslant \lambda) \leqslant e^{-a\lambda}E(e^{aX})$;
(b) $\mathcal{P}(|X| \geqslant \lambda) \leqslant e^{-a\lambda}\{E(e^{aX}) + E(e^{-aX})\}$.

**Proof.**

$$E(e^{aX}) = \int_{X \geqslant \lambda} e^{aX}\,d\mathcal{P} + \int_{X<\lambda} e^{aX}\,d\mathcal{P} \geqslant \mathcal{P}(X \geqslant \lambda)e^{a\lambda}. \tag{1}$$

This gives (a).

$$E(e^{-aX}) = \int_{X \leqslant -\lambda} e^{-aX}\,d\mathcal{P} + \int_{X>-\lambda} e^{-aX}\,d\mathcal{P} \geqslant \mathcal{P}(X \leqslant -\lambda)e^{(-a)(-\lambda)}.$$

Hence

$$\mathcal{P}(X \leqslant -\lambda) \leqslant e^{-a\lambda}E(e^{-aX}). \tag{2}$$

Combining (1) and (2), we have

$$\mathcal{P}\big(|X| \geqslant \lambda\big) = \mathcal{P}(X \leqslant -\lambda) + \mathcal{P}(X \geqslant \lambda) \leqslant e^{-a\lambda}E(e^{-aX}) + e^{-a\lambda}E(e^{aX}). \qquad \square$$

**Lemma 2.3.** *Let $N > 0$ and let $Y_1, \ldots, Y_q$ be random variables such that $E(e^{\lambda b Y_k}) \leqslant e^{Nb^2/2}$ for all real $b$ and $k = 1, \ldots, q$. Then*

$$\mathcal{P}\big(|Y_k| < \sqrt{2N\log 4q},\ \textit{for all } k = 1, \ldots, q\big) > \frac{1}{2}.$$

**Proof.** By Lemma 2.2, given $\lambda \geqslant 0$,

$$\mathcal{P}\big(|Y_k| \geqslant \lambda,\ \text{for some } k = 1, \ldots, q\big) \leqslant \sum_{k=1}^{q} \mathcal{P}\big(|Y_k| \geqslant \lambda\big)$$
$$\leqslant e^{-a\lambda}\left[\sum_{k=1}^{q}\big(E(e^{aY_k}) + E(e^{-aY_k})\big)\right]$$

for any $a \geqslant 0$. By hypothesis, this last expression is less than

$$e^{-a\lambda} 2q e^{Na^2/2}. \tag{3}$$

Setting $a = \lambda/N$ and then setting $\lambda = \sqrt{2N \log 4q}$, (3) becomes $1/2$. Thus,

$$\mathcal{P}\big(|Y_k| < \sqrt{2N \log 4q},\ \text{for all } k = 1, \ldots, q\big) \geqslant \frac{1}{2}. \qquad \square$$

The next lemma follows from elementary arguments and the mean value theorem.

**Lemma 2.4.** *Let* $0 \leqslant \alpha \leqslant 1$, $-1 < \gamma \leqslant 0$, *and* $u_k = [e^{k^\alpha}]$, *where* $[\cdot]$ *denotes the greatest whole integer function. There exist constants* $A, C_1, C_2, C_3, C_4 > 0$ (*depending only on* $\alpha$ *and* $\gamma$) *such that for* $k \geqslant A$

(a) $u_{k+1} > u_k$;
(b) $C_1 k^{\alpha-1} e^{k^\alpha} \leqslant u_k - u_{k-1} \leqslant C_2 k^{\alpha-1} e^{k^\alpha}$;
(c) $C_3 k^{\gamma-1} e^{-k^\alpha} \leqslant \frac{(k-A)^\gamma - (k-A-1)^\gamma}{u_k - \mu_{k-1}} - \frac{(k-A+1)^\gamma - (k-A)^\gamma}{u_{k+1} - u_k} \leqslant C_4 k^{\gamma-1} e^{-k^\alpha}$.

**Lemma 2.5.** *For* $\frac{1}{3} \leqslant \alpha \leqslant 1$ *and* $-1 \leqslant \beta \leqslant 0$ *there exists a constant* $C$ *depending only on* $\alpha$ *and* $\beta$ *such that*

$$\sum_{k=1}^{N} k^\beta e^{k^\alpha} \leqslant C N^{1+\beta-\alpha} e^{N^\alpha}.$$

**Proof.** Since $x^\beta e^{x^\alpha}$ is monotonic for $x > 27$, we can estimate the sum by an integral of the form $\int_0^N x^\beta e^{x^\alpha}\, dx$. Change of variables, integration by parts and elementary estimations give the result. $\square$

**Lemma 2.6.** *Let* $m_k$, $k = 0, \ldots, N$, *denote integers such that* $0 < m_0 < m_1 < \cdots < m_N$, *and let* $r_k$ *denote real numbers such that* $r_1 > \cdots > r_N > 0$. *Let* $a_n = r_k$ *for* $m_{k-1} \leqslant n < m_k$ *for* $1 \leqslant n \leqslant N$, *and let* $a_n = 0$ *otherwise. Define intervals* $I_0 = (\pi/m_1, \pi]$, $I_j = (\pi/m_{j+1}, \pi/m_j]$ *for* $j = 1, \ldots, N-1$, *and* $I_N = [0, \pi/m_{N-1})$. *For* $x \in I_j$

$$\left|\sum_{n\in\mathbf{Z}} a_n \sin nx\right| \leqslant \frac{\pi^2}{4} \frac{1}{m_j} \sum_{k=1}^{j} (r_k - r_{k+1}) m_k^2 + r_{j+1} m_{j+1}$$

*with the following conventions*:

(a) *If a lower index exceeds an upper index in a summation, the sum is void*;
(b) $r_{N+1} m_{N+1}$ *is defined to be* $0$.

**Proof.** Let

$$S(x) = \sum_{n \in \mathbf{Z}} a_n \sin nx = \sum_{k=1}^{N} r_k \sum_{j=m_{k-1}}^{m_k - 1} \sin jx.$$

Defining $r_{N+1}$ to be 0, this can be rewritten as

$$\begin{aligned} S(x) &= \sum_{k=1}^{N} (r_k - r_{k+1}) \sum_{j=m_0}^{m_k - 1} \sin jx \\ &= \frac{1}{\sin x/2} \sum_{k=1}^{N} (r_k - r_{k+1}) \sin \frac{(m_k + m_0 - 1)x}{2} \sin \frac{(m_k - m_0)x}{2}. \end{aligned}$$

Now suppose $(\pi/m_{j+1}) < x \leqslant (\pi/m_j)$ where $1 \leqslant j \leqslant N-1$. We write $S(x) = S_1(x) + S_2(x)$ where $S_1(x)$ is the sum over $1 \leqslant k \leqslant j$ and $S_2(x)$ is the sum over $j+1 \leqslant k \leqslant N$. For such $x$

$$\begin{aligned} &\frac{1}{\sin x/2} \left| \sin \frac{(m_k + m_0 - 1)x}{2} \sin \frac{(m_k - m_0)x}{2} \right| \\ &\quad \leqslant (m_k + m_0 - 1)(m_k - m_0) x^2 4 \sin(x/2) \leqslant \bigl(m_k(m_k - 1) - m_0(m_0 - 1)\bigr) \frac{\pi^2}{4m_j}. \end{aligned}$$

Hence

$$\bigl|S_1(x)\bigr| \leqslant \frac{\pi^2}{4m_j} \sum_{k=1}^{j} (r_k - r_{k+1}) m_k (m_k - 1) - \frac{\pi^2}{4m_j} (r_1 - r_{j+1}) m_0 (m_0 - 1).$$

Also

$$\bigl|S_2(x)\bigr| \leqslant \frac{1}{\sin(x/2)} \sum_{k=j+1}^{N} (r_k - r_{k+1}) = \frac{1}{\sin(x/2)} r_{j+1} \leqslant \frac{\pi}{x} r_{j+1} \leqslant r_{j+1} m_{j+1}.$$

This gives for $(\pi/m_{j+1}) < x \leqslant (\pi/m_j)$ and $0 \leqslant j \leqslant N-1$

$$\begin{aligned} \bigl|S(x)\bigr| \leqslant{}& \frac{\pi^2}{4} \left( \frac{1}{m_j} \sum_{k=1}^{j} (r_k - r_{k+1}) m_k (m_k - 1) - \frac{m_0(m_0 - 1)}{m_j} (r_1 - r_{j+1}) \right) \\ &+ r_{j+1} m_{j+1}. \end{aligned}$$

If $0 \leqslant x \leqslant (\pi/m_N)$ then $S(x) = S_1(x)$ and the above estimates for $S_1(x)$ still apply. Therefore, the last estimate for $S(x)$ is valid for $j = N$ where $r_{N+1} = 0$ so that $r_{N+1} m_{N+1} = 0$ even though $m_{N+1}$ has not been defined. If $(\pi/m_1) < x < \pi$ then $S(x) = S_2(x)$ and we get $|S(x)| \leqslant r_1 m_1$. Hence we complete the proof of the lemma. □

**Lemma 2.7.** *Let* $\frac{1}{3} < \alpha \leqslant 1$. *Let* $d \geqslant 1$ *and* $N > 1$ *be integers. Let* $A \geqslant 0$ *and define* $m_k = [e^{(A+k)^\alpha}]$ *for* $k = 0, 1, \ldots, N$, *where* $[\cdot]$ *denotes the greatest whole integer function. Suppose* $A$ *is large enough to give* $m_0 < m_1 < \cdots < m_N$. *Define*

$$r_k = \frac{k^{1/d} - (k-1)^{1/d}}{m_k - m_{k-1}}$$

*and*

$$a_n = \begin{cases} r_k, & m_{k-1} \leqslant n < m_k,\ k = 1, \ldots, N, \\ 0, & n < m_0 \text{ or } n \geqslant m_N. \end{cases}$$

*Then there exists constants $A$ and $C$ such that for all $x$*

$$\left| \sum_{n \in \mathbf{Z}} a_n \sin nx \right| \leqslant C N^{(1/d)-\alpha}.$$

**Proof.** Let $c_k = r_k - r_{k+1}$. We will apply Lemmas 2.4–2.6. As in Lemma 2.4, let $u_k$ denote $[e^{k^\alpha}]$. First consider

$$\frac{1}{m_j} \sum_{k=1}^{j} c_k m_k^2 = \frac{1}{u_{A+j}} \sum_{k=A+1}^{A+j} \left[ \frac{k^{1/d} - (k-1)^{1/d}}{u_k - u_{k-1}} - \frac{(k+1)^{1/d} - k^{1/d}}{u_{k+1} - u_k} \right] u_k^2.$$

Setting $\gamma = 1/d$ in Lemma 2.4 and $\beta = (1/d) - 1$ in Lemma 2.5 gives

$$\frac{1}{m_j} \sum_{k=1}^{j} c_k m_k^2 \leqslant C' e^{-(A+j)^\alpha} \sum_{k=A+1}^{A+j} k^{(1/d)-1} e^{k^\alpha} \leqslant C''(A+j)^{(1/d)-\alpha} \leqslant C''' N^{(1/d)-\alpha},$$

where $C'''$ depends only on $\alpha$ and $d$. Next, if $1 \leqslant q \leqslant N$, Lemma 2.4 gives

$$\begin{aligned} r_q m_q &= \left( \frac{q^{1/d} - (q-1)^{1/d}}{m_q - m_{q-1}} \right) m_q \leqslant D \left( \frac{q^{(1/d)-1}}{(A+q)^{\alpha-1} e^{(A+q)^\alpha}} \right) e^{(A+q)^\alpha} \\ &\leqslant D'(A+q)^{(1/d)-1} \leqslant D' N^{(1/d)-1}, \end{aligned}$$

where $D'$ depends only on $\alpha$ and $d$. Inserting these estimates into Lemma 2.6 gives the result. □

The following lemma is well known.

**Lemma 2.8.** *Let $f(x) = \sum_k a_k e^{ikx}$ be a real trigonometric polynomial on $[0, 2\pi)^d$. Suppose*

$$\sup\{|k_1|, \ldots, |k_d|\} \leqslant M.$$

*Then there exist points $x_1, \ldots, x_q$ where $q < CM^d$ and $C$ depends only on $d$, such that*

$$\|f\|_\infty \leqslant 2 \max_{j=1,\ldots,q} \left| f(x_j) \right|.$$

**Proof.** Choose $z$ such that $|f(z)| = \|f\|_\infty$. Let $K$ be an integer not less than $4\pi dM$, and let the $x_j$ be the lattice points in $[0, 2\pi)^d$ with coordinate spacing equal to $2\pi/K$. There are $q = K^d$ such points. Consider the $x_j$ that is closest to $z$. Then by the mean value theorem there is a $c_j$ on the line segment from $z$ to $x_j$ such that

$$f(z) - f(x_j) = \sum_{q=1}^{d} \frac{\partial f}{\partial t_q}(c_j)(z_q - x_{jq}).$$

Therefore

$$\left|f(z) - f(x_j)\right| \leqslant \sum_{q=1}^{d} \left|\frac{\partial f}{\partial t_q}(c_j)\right| \max_{r=1,\dots,d} \left|(z_r - x_{jr})\right|.$$

Define $\epsilon_q = \operatorname{sgn} \partial/\partial t_q(c_j)$, and define $u(\theta) = f(c_{j1} + \epsilon_1\theta, \dots, c_{jd} + \epsilon_d\theta)$. Then $u$ is a single-variable real trigonometric polynomial of degree $Md$ and

$$u'(0) = \sum_{q=1}^{d} \left|\frac{\partial f}{\partial t_q}(c_j)\right|.$$

By Bernstein's theorem $|u'(0)| \leqslant Md\|u\|_\infty \leqslant Md\|f\|_\infty$. Hence

$$\left|f(z) - f(x_j)\right| \leqslant Md\|f\|_\infty(2\pi/K) \leqslant \|f\|_\infty/2$$

and the conclusion follows. □

Before stating the main theorem, we establish some notation. Fix integers $N > 1$ and $d \geqslant 1$. For $k = 1, \dots, N$ define $m_k$ and $r_k$ as in Lemma 2.7 (we will assign values to $\alpha$ and $A$ later). Let

$$I_k = [m_k, m_{k-1}) \cap \mathbf{Z}, \qquad J_k = \bigcup_{j=1}^{k} I_j, \qquad B_k = J_k^d \subset \mathbf{Z}^d,$$
$$L_1 = B_1, \qquad L_k = B_k \setminus B_{k-1} \quad \text{for } k = 2, \dots, N.$$

On $J_N$ define the discrete measure $\nu$ by $\nu(n) = r_k$ for $n \in I_k$. Then $\nu(I_k) = k^{1/d} - (k-1)^{1/d}$ and $\nu(J_k) = k^{1/d}$. Define a measure $\nu_k$ on $J_k$ by the restriction of $\nu$ to $J_k$. Define the measure $\mu_k$ on $B_k^d$ by the product $\mu_k = \nu_k^d$. Then $\mu_k(B_k) = k$ and $\mu_k(L_k) = 1$. Define the probability measure $P_k$ on $L_k$ by the restriction of $\mu_k$ to $L_k$.

**Theorem 2.9.** *Given integers $N > 1$ and $d \geqslant 1$, setting $\alpha = 1/(1+2d)$ and using the notation above, there exists constants $A > 0$ and $C$ depending only on $d$ such that if the multi-integers $\lambda_1, \dots, \lambda_N$ are chosen at random, independently and uniformly from the sets $L_1, \dots, L_N$, then the probability that*

$$\left|\sum_{k=1}^{N} \sin\lambda_{k1}x_1 \dots \sin\lambda_{kd}x_d\right| \leqslant CN^{(d+1)/(2d+1)}$$

*is not less than* $1/2$.

**Proof.** Let $\Lambda_1, \dots, \Lambda_N$ be independent random variables where $\Lambda_k$ is uniformly distributed on $L_k$. Define for $x = (x_1, \dots, x_d)$ and $\lambda = (\lambda_1, \dots, \lambda_d)$

$$S(x, \lambda) = \prod_{j=1}^{d} \sin x_j\lambda_j.$$

Let $f$ be the random trigonometric polynomial in $[0, 2\pi)^d$ defined by

$$f(x) = \sum_{k=1}^{N} S(x, \Lambda_k).$$

Note that the degree $M$ of $f$ is less than $dm_N$. Let $x_j$ for $j = 1, \dots, q$ be the lattice points in $[0, 2\pi)^d$ as defined in Lemma 2.8. We will estimate the expectation of the random variables $Z_j = f(x_j)$:

$$E(Z_j) = \sum_{k=1}^{N} E\big(S(x_j, \Lambda_k)\big) = \sum_{k=1}^{N} \sum_{\lambda_k \in L_k} S(x_j, \lambda_k)\mu_k(\lambda_k).$$

Since $\mu_k(\lambda_k) = \prod_{j=1}^{d} \nu(\lambda_{kj})$,

$$\begin{aligned} E(Z_j) &= \sum_{k=1}^{N} \sum_{\lambda_k \in L_k} \left( \prod_{j=1}^{d} \sin \lambda_{kj} x_j \right) \left( \prod_{j=1}^{d} \nu(\lambda_{kj}) \right) \\ &= \sum_{\lambda \in B_N} \prod_{j=1}^{d} \sin(\lambda_j x_j)\big(\nu(\lambda_j)\big) = \left( \sum_{\lambda \in J_N} \nu(\lambda) \sin \lambda x \right)^d. \end{aligned}$$

Recalling that $\alpha = 1/(1+2d)$ and using Lemma 2.7 gives $E(Z_j) \leqslant CN^{(d+1)/(2d+1)}$. Next, by Lemma 2.1 we have

$$\begin{aligned} E\big(\exp\big[\lambda\big(Z_j - E(Z_j)\big)\big]\big) &= E\left( \exp\left( \lambda \sum_{k=1}^{N} S(x_j, \Lambda_k) - E\big(S(x_j, \Lambda_k)\big) \right) \right) \\ &= \prod_{j=1}^{N} E\Big( \exp\big( \lambda\big[ S(x_j, \Lambda_k) - E\big(S(x_j, \Lambda_k)\big)\big]\big)\Big) \\ &\leqslant \prod_{k=1}^{N} \exp(\lambda^2/2) = \exp(N\lambda^2/2). \end{aligned}$$

Therefore, by Lemma 2.3,

$$\mathcal{P}\Big( \big|Z_j - E(Z_j)\big| \leqslant \sqrt{2N \log 4q}, \text{ for } j = 1, \dots, q \Big) \geqslant \frac{1}{2}$$

and

$$\mathcal{P}\Big( \max_{j=1,\dots,q} \big|f(x_j)\big| \leqslant \big|E(Z_j)\big| + \sqrt{2N \log 4q} \Big) \geqslant \frac{1}{2}.$$

By Lemma 2.8, $\|f\|_\infty \leqslant C' \max_{j=1,\dots,q} |f(x_j)|$. Also $q \leqslant C'' e^{N^\alpha} = e^{N^{1/(2d+1)}}$ and $|E(Z_j)| \leqslant C''' N^{(d+1)/(2d+1)}$. Therefore there exists a constant $C$ such that

$$\mathcal{P}\big( \|f\|_\infty \leqslant CN^{(d+1)/(2d+1)} \big) \geqslant \frac{1}{2}.$$

We remark that the probability $1/2$ can be replaced by probability $1 - \epsilon$ for any $\epsilon > 0$, by choosing a larger ($\epsilon$-dependent) constant $C$. $\square$

## References

[1] J. Bourgain, Sur les sommes de sinus, Sem. Anal. Harm. Publ. Math. d'Orsay 84-01 (1984), exp. no. 3.
[2] S.V. Konyagin, Estimates of maxima of sine sums, East. J. Approx. 3 (1997) 63–70.